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ON SOLVING HAMILTON'S GENERAL QUADRATIC EQUATION

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ABSTRACT

Based on *Hamilton*'s method of solving the quadratic equation (couple equation), defined over the field of couple numbers, as he calls them, in this paper the method of solving the general Hamilton quadratic equation is reduced to solving of the matrix quadratic equation, which is defined over the field of bireal matrices, which correspond to bireal numbers. As the matrix equation is decomposed into a system of nonlinear algebraic equations (SNAE), the solutions of the matrix equation are also the solutions of the SNAE, such that the solutions of the matrix quadratic equation implicitly determine the symmetric matrices S of the null space of the SNAE. In the second part of the paper, the matrix method for solving the SNAE, which is obtained via decomposition of *Hamilton*'s quadratic equation

$$(a_{11},a_{21})(x,0)^2 - (a_{12},a_{22})(0,y)^2 + 2(b_2,-b_1)(x,0)(0,y) +$$
$$+ (c_{11},c_{21})(x,0) + (c_{22},-c_{12})(0,y) + (d_1,d_2) = (0,0).$$

Keywords: bireal numbers, couple equation, matrix quadratic equation.

INTRODUCTION

In the middle of the 19th century, on the basis of the previously defined product of ordered pairs of real numbers (number couples) (Hamilton, 1837, Hamilton, 1853), as follows

$$(a,b)(c,d) = (ac-bd,ad+bc), \qquad (1)$$

the process of solving the quadratic equation (couple equation)

$$(x,y)^2 + (b,0)(x,y) + (c,0) = (0,0), \qquad (2)$$

Hamilton reduced to solving an ordinary quadratic equation $x^2 + bx + c = 0$. More precisely, *Hamilton* reduced the process of solving the equation (1) to the process of solving a system of two separate equations

$$x^2 - y^2 + bx + c = 0 \text{ and } 2xy + by = 0, \qquad (3)$$

which always allows real solutions, regardless of whether the discriminant $b^2/4 - c$ is a positive or negative real number, (Hamilton, 1853). In the first case, $y = 0$ in the second equation of the system, which leads us to the first solution

$$(x,y) = (-\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}, 0). \qquad (4)$$

In the second case, the second factor $2x + b$, in the second equation of system (3), should be equalized to zero and thus reach another solution

$$(x,y) = (-\frac{b}{2}, \pm\sqrt{c - \frac{b^2}{4}}). \quad (5)$$

On the other hand, systems of nonlinear algebraic equations (SNAE), to which system (3) itself belongs, are more ubiquitous in many numerous applications. Solving SNAE, in general, requires numerical simulation, and increasingly robust and efficient methods for solving SNAE are continuously sought. At the end of the last century, as well as at the beginning of this century, the so-called method of neural networks was imposed, (Goulianas et all,, 2013, Li & Zhezhao, 2008, Margaris & Goulianas, 2012). In (Borisevich et all., 2000) an overview of existing algorithmic approaches for solving SNAE: reduction to *Grebner's* basis, the multidimensional resulting method and the spectral method. In addition, it is shown that the problem of solving SNAE is equivalent to the problem of finding a rank 1 matrix in a given subspace of a space of matrices. All of these methods start from a system of k nonlinear algebraic equations ($k \geq 1$) with n unknowns x_i, which are elements of the $n \times 1$ column matrix $\mathbf{x} = (x_i)^{\mathrm{T}} = [x_1\ x_2\ ...\ x_n]^{\mathrm{T}}$ ($n \geq 2$), as follows

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij})_k x_i x_j + 2\sum_{i=1}^{n}(b_i)_k x_i + c_k = 0. \quad (6)$$

In the matrix notation of system (6)

$$\mathbf{x}^{\mathrm{T}} A_k \mathbf{x} + 2\mathbf{b}_k^{\mathrm{T}}\mathbf{x} + c_k = 0, \quad (7)$$

the $n \times n$ matrices $A_k = (a_{ij})_k$ are symmetric matrices, $\mathbf{b}_k^{\mathrm{T}} = (b_i)_k^{\mathrm{T}}$ are $1 \times n$ row matrices and c_k are constants. Let $\hat{A}_k$ denote the accompanying square symmetric matrices of higher order by 1 of the matrices A_k, obtained by bordering the matrices A_k on the right and from below by the column matrices $(\mathbf{b}_k^{\mathrm{T}}, c_k)^{\mathrm{T}}$ and the row matrices $(\mathbf{b}_k^{\mathrm{T}}, c_k)$. The null space of system (6) is a set L_0 of all symmetric square matrices S of order $n + 1$ satisfying a system of equalities for traces

$$\mathrm{tr}(\hat{A}_k S) = 0. \quad (8)$$

Theorem 1. which follows, and is taken from (Borisevich et all., 2000), allows for simplification in the solving of some concrete systems of type (6).

Teorem 1. *For each ordered set of numbers $s_1, s_2, ..., s_n$, the system matrix S may be formed as* $S = [s_1\ s_2\ ...\ s_n\ 1]^{\mathrm{T}}[s_1\ s_2\ ...\ s_n\ 1]$ *that provides the one-to-one correspondence between all solutions of system* (6) *and the set of all affine matrices of rank* 1 *of zero-space of system* (6).

As emphasized in (Borisevich et all., 2000), for the practical use of *Theorem* 1., it is enough to determine a basis $E_1, E_2, ..., E_q$ of the null space L_0 of the system (6). The problem of solving system (6) is equivalent to the problem of searching all such numbers $\alpha_1, \alpha_2, ..., \alpha_q$, for which the matrices $\alpha_1 E_1 + \alpha_1 E_1 + ... + \alpha_q E_q$ are affine matrices of rank 1.

In (Bardell, 2014) and (Hardy, 2008), an explicit (qualitative) analysis of solutions of SNAE (6), in the case where $n = k = 2$, was presented. This type of system (6) is obtained via decomposition of *Hamilton's* general quadratic equation

$$(a_1, a_2)(x, y)^2 + (b_1, b_2)(x, y) + (c_1, c_2) = (0,0). \quad (9)$$

This paper brings an implicit method for solving this type of SNAE (5), which is based on solving a matrix quadratic equation

$$AX^2 + BX + C = O, \qquad (10)$$

where $A = \begin{bmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & b_2 \\ -b_2 & b_1 \end{bmatrix}$, $C = \begin{bmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{bmatrix}$, $X = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$ and O is a square zero matrix. The matrix quadratic equation (10) corresponds to the quadratic equation (9). In the second part of the paper, the matrix method for solving the SNAE is presented, which is obtained via decomposition of the following *Hamilton's* quadratic equation (couple equation)

$$\begin{aligned}&(a_{11},a_{21})(x,0)^2 - (a_{12},a_{22})(0,y)^2 + 2(b_2,-b_1)(x,0)(0,y) + \\ &+ (c_{11},c_{21})(x,0) + (c_{22},-c_{12})(0,y) + (d_1,d_2) = (0,0). \qquad (11)\end{aligned}$$

SOLUTIONS OF HAMILTON'S GENERAL QUADRATIC EQUATION

Solutions of quadratic equations $ax^2 + bx + c = 0$, over the field of real numbers $\mathbb{R}$, exist if the discriminant of the system $\Delta = b^2 - 4ac$ is nonnegative. On the other hand, the elements w of the field $\mathbb{R}^2$ (the *Cartesian* square of the set of real numbers $\mathbb{R}$) are ordered pairs (x,y) of real numbers (*Hamilton's* number couples), which we can rename bireal numbers. As $(x,y) = x(1,0) + y(0,1)$, where $(1,0)$ and $(0,1)$ are the basis elements of the field of bireal numbers $\mathbb{R}^2$ (Hamilton, 1837), the quadratic equation, over the field of bireal numbers $\mathbb{R}^2$,

$$(a_1,a_2)w^2 + (b_1,b_2)w + (c_1,c_2) = (0,0), \qquad (12)$$

where $w = (x,y)$, can be decomposed, such that

$$\begin{aligned}(a_1,a_2)w^2 + (b_1,b_2)w + (c_1,c_2) &= a_1w^2 + b_1w + (c_1,0) + a_2ww_\perp + b_2w_\perp + (0,c_2) = \\ &= [a_1w^2 + b_1w + (c_1,0)](1,0) + [a_2w^2 + b_2w + (c_2,0)](0,1) = (0,0), \end{aligned} \qquad (13)$$

where $w_\perp = (0,1)w$ is the bireal number $(-y,x)$, which is obtained by rotating the bireal number w, by an angle of $\pi/2$ radians, in the positive mathematical direction.

Between the elements w of the field of bireal numbers $\mathbb{R}^2$ and square matrices of the second order $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}$, which are the elements of the field of bireal matrices R^2, whose basis consists of the identity matrix $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the antisymmetric matrix $\hat{E} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, there is a one-to-one correspondence. Accordingly, the product of bireal numbers (1) corresponds to the product of the corresponding bireal matrices and vice versa

$$(a,b)(c,d) \leftrightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{bmatrix} \leftrightarrow (ac-bd, ad+bc). \qquad (14)$$

The inverse bireal number w^{-1} ($w \neq (0,0)$) corresponds to the inverse matrix $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}^{-1}$, so that

$$w^{-1} = \frac{\bar{w}}{\|w\|^2} \quad \frac{1}{x^2+y^2}\begin{bmatrix} x & -y \\ y & x \end{bmatrix}, \tag{15}$$

where $\|w\| = (x^2 + y^2)^{1/2}$ is the *Euclidean* norm of $\mathbb{R}^2$ and $\bar{w} = (x, -y)$. Since $X = xE + y\hat{E}$, *Hamilton's* the general quadratic equation (9), in matrix form, obtained from the matrix equation (10), is as follows

$$AX^2 + BX + C = \begin{bmatrix} a_1(x^2-y^2) - 2a_2xy & a_2(x^2-y^2) + 2a_1xy \\ -[a_2(x^2-y^2) + 2a_1xy] & a_1(x^2-y^2) - 2a_2xy \end{bmatrix} + \begin{bmatrix} b_1x - b_2y & b_1y + b_2x \\ -(b_1y + b_2x) & b_1x - b_2y \end{bmatrix} +$$

$$+\begin{bmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{bmatrix} = a_1\begin{bmatrix} x^2-y^2 & 2xy \\ -2xy & x^2-y^2 \end{bmatrix} + b_1\begin{bmatrix} x & y \\ -y & x \end{bmatrix} + c_1E +$$
$$+a_2\begin{bmatrix} -2xy & x^2-y^2 \\ -(x^2-y^2) & -2xy \end{bmatrix} + b_2\begin{bmatrix} -y & x \\ -x & -y \end{bmatrix} + c_1\hat{E} = O, \tag{16}$$

So that $AX^2 + BX + C = (a_1X^2 + b_1X + c_1E)E + (a_2X^2 + b_2X + c_2E)\hat{E} = O.$ According to (13), that is (16), the field of bireal matrices $\{X : \hat{E}Y_1 = Y_2\} \subset \mathrm{R}^2$, where Y_i: $\mathrm{R}^2 \to \mathrm{R}^2$ (i = 1,2) are square matrix functions $a_iX^2 + b_iX + c_iE$, is the null field L_0 (the set of solutions) of the matrix quadratic equation (16). Obviously, according to *Theorem* 1., the real numbers $s_1 = x$ and $s_2 = y$, where x and y are elements of the bireal matrices $X \in L_0$, are elements of the matrices $[s_1\ s_2\ 1]$, such that the matrices $S = [s_1\ s_2\ 1]^{\mathrm{T}}[s_1\ s_2\ 1]$ are symmetric matrices of the null space of SNAE

$$\begin{gathered}\sum_{i=1}^{2}\sum_{j=1}^{2}(a_{ij})_1x_ix_j + 2\sum_{i=1}^{2}(b_i)_1x_i + c_1 = 0 \\ \sum_{i=1}^{2}\sum_{j=1}^{2}(a_{ij})_2x_ix_j + 2\sum_{i=1}^{2}(b_i)_2x_i + c_2 = 0,\end{gathered} \tag{17}$$

which can be written in a more concise form

$$\mathbf{x}^{\mathrm{T}}A_1\mathbf{x} + 2\mathbf{b}_1^{\mathrm{T}}\mathbf{x} + \frac{c_1}{a_1} = 0 \quad \mathbf{x}^{\mathrm{T}}A_2\mathbf{x} + 2\mathbf{b}_2^{\mathrm{T}}\mathbf{x} + \frac{c_2}{a_2} = 0, \tag{18}$$

where $\mathbf{x}^{\mathrm{T}} = [x_1\ x_2] = [x\ y]$,

$$A_1=\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} \end{bmatrix}=\begin{bmatrix} 1 & -\dfrac{a_2}{a_1} \\ -\dfrac{a_2}{a_1} & -1 \end{bmatrix},\ A_2=\begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} \\ a_{21}^{(2)} & a_{22}^{(2)} \end{bmatrix}=\begin{bmatrix} 1 & \dfrac{a_1}{a_2} \\ \dfrac{a_1}{a_2} & -1 \end{bmatrix},$$

$$\mathbf{b}_1^{\mathrm{T}}=\begin{bmatrix} b_1^{(1)} & b_2^{(1)} \end{bmatrix}=\begin{bmatrix} \dfrac{b_1}{2a_1} & -\dfrac{b_2}{2a_1} \end{bmatrix} \text{ and } \mathbf{b}_2^{\mathrm{T}}=\begin{bmatrix} b_1^{(2)} & b_2^{(2)} \end{bmatrix}=\begin{bmatrix} \dfrac{b_2}{2a_2} & \dfrac{b_1}{2a_2} \end{bmatrix}.$$

By (8), matrices S are affine matrices of rank 1, which satisfy a system of equalities for traces

$$\mathrm{tr}(\hat{A}_1S)=0 \text{ and } \mathrm{tr}(\hat{A}_2S)=0, \qquad (19)$$

$$\text{where } \hat{A}_1=\begin{bmatrix} 1 & -\dfrac{a_2}{a_1} & \dfrac{b_1}{2a_1} \\ -\dfrac{a_2}{a_1} & =1 & -\dfrac{b_2}{2a_1} \\ \dfrac{b_1}{2a_1} & -\dfrac{b_2}{2a_1} & \dfrac{c_1}{a_1} \end{bmatrix} \text{ and } \hat{A}_2=\begin{bmatrix} 1 & \dfrac{a_1}{a_2} & \dfrac{b_2}{2a_2} \\ \dfrac{a_1}{a_2} & =1 & \dfrac{b_1}{2a_2} \\ \dfrac{b_2}{2a_2} & \dfrac{b_1}{2a_2} & \dfrac{c_2}{a_2} \end{bmatrix}.$$

On the other hand, the solutions of system (17) are the solutions of matrix equation (16). Since bireal matrices, if they are not zero matrices, are regular and commutative matrices, bireal matrices

$$X_1=-\frac{A^{-1}B}{2}+\sqrt{(\tfrac{A^{-1}B}{2})^2-A^{-1}C} \text{ and } X_2=-\frac{A^{-1}B}{2}-\sqrt{(\tfrac{A^{-1}B}{2})^2-A^{-1}C}, \qquad (20)$$

are solutions of the matrix equation (16). Considering the fact that

$$A^{-1}B=\frac{1}{a_1^2+a_2^2}\begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix}\begin{bmatrix} b_1 & b_2 \\ -b_2 & b_1 \end{bmatrix}=\frac{1}{a_1^2+a_2^2}\begin{bmatrix} a_1b_1+a_2b_2 & a_1b_2-a_2b_1 \\ a_2b_1-a_1b_2 & a_1b_1+a_2b_2 \end{bmatrix},$$

$$A^{-1}C=\frac{1}{a_1^2+a_2^2}\begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix}\begin{bmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{bmatrix}=\frac{1}{a_1^2+a_2^2}\begin{bmatrix} a_1c_1+a_2c_2 & a_1c_2-a_2c_1 \\ a_2c_1-a_1c_2 & a_1c_1+a_2c_2 \end{bmatrix}$$

$$\text{and } (A^{-1}B)^2=\frac{1}{(a_1^2+a_2^2)^2}\begin{bmatrix} (a_1^2-a_2^2)(b_1^2-b_2^2)+4a_1a_2b_1b_2 & 2[(a_1^2-a_2^2)b_1b_2-(b_1^2-b_2^2)a_1a_2] \\ 2[(b_1^2-b_2^2)a_1a_2-(a_1^2-a_2^2)b_1b_2] & (a_1^2-a_2^2)(b_1^2-b_2^2)+4a_1a_2b_1b_2 \end{bmatrix},$$

the bireal matrix $D^2=\begin{bmatrix} d_1^2-d_2^2 & 2d_1d_2 \\ -2d_1d_2 & d_1^2-d_2^2 \end{bmatrix}$, whose elements satisfy SNAE

$$d_1^2 - d_1^2 = \frac{(a_1b_1 + a_2b_2)^2 - (a_1b_2 - a_2b_1)^2 - 4(a_1^2 + a_2^2)(a_1c_1 + a_2c_2)}{4(a_1^2 + a_2^2)^2}$$
$$2d_1d_2 = \frac{2(a_1b_1 + a_2b_2)(a_1b_2 - a_2b_1) - 4(a_1^2 + a_2^2)(a_1c_2 - a_2c_1)}{4(a_1^2 + a_2^2)^2}, \tag{21}$$

is the matrix discriminant of the matrix equation (16). SNAE (21) can be solved graphically and analytically. Clearly, the graphic solution is the coordinates of the points of intersection of two second order curves, as shown in Figure1

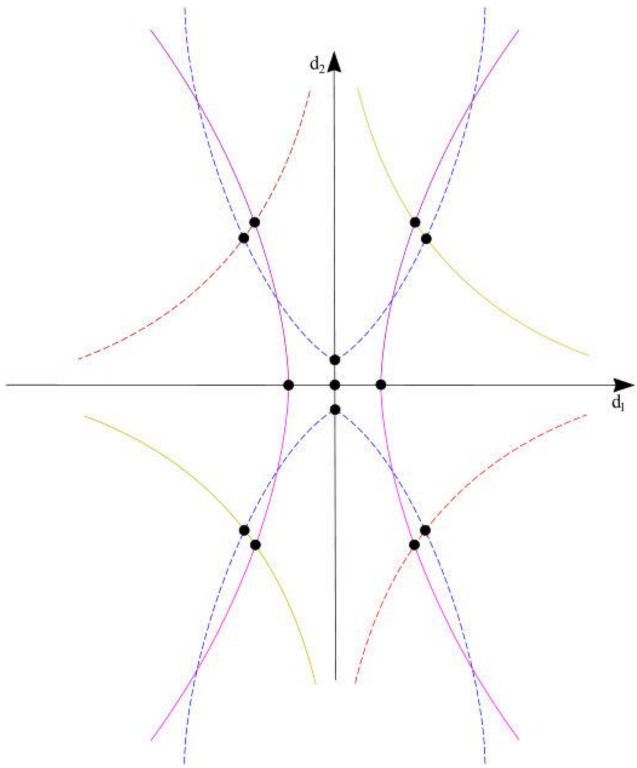

Figure 1.

On the other hand, the analytical solution can be obtained by converting *Cartesian* coordinates into polar coordinates

$$d_1 = r\cos\varphi \text{ and } d_2 = r\sin\varphi. \tag{22}$$

Therefore, in polar coordinates

$$r^2\cos 2\varphi = \frac{(a_1b_1 + a_2b_2)^2 - (a_1b_2 - a_2b_1)^2 - 4(a_1^2 + a_2^2)(a_1c_1 + a_2c_2)}{4(a_1^2 + a_2^2)} \text{ and}$$
$$r^2\sin 2\varphi = \frac{2(a_1b_1 + a_2b_2)(a_1b_2 - a_2b_1) - 4(a_1^2 + a_2^2)(a_1c_2 - a_2c_1)}{4(a_1^2 + a_2^2)}. \tag{23}$$

tath is, $\tan 2\varphi = \dfrac{2(a_1b_1+a_2b_2)(a_1b_2-a_2b_1)-4(a_1^2+a_2^2)(a_1c_2-a_2c_1)}{(a_1b_1+a_2b_2)^2-(a_1b_2-a_2b_1)^2-4(a_1^2+a_2^2)(a_1c_1+a_2c_2)}$.

Finally, after determining the elements d_1 and d_2 of the matrix $D=\begin{bmatrix} d_1 & d_2 \\ -d_2 & d_1 \end{bmatrix}$, the matrices $X_{1,2}=-\dfrac{A^{-1}B}{2}\pm D$ are solutions of the matrix equation (16). On the basis of (21), it can be concluded that if

$$\begin{aligned}&(a_1b_1+a_2b_2)(a_2b_1-a_1b_2)=2(a_1^2+a_2^2)(a_1c_2-a_2c_1) \text{ and}\\ &(a_1b_1+a_2b_2)^2-(a_2b_1-a_1b_2)^2\geq 4(a_1^2+a_2^2)(a_1c_1+a_2c_2),\end{aligned} \tag{24}$$

then the solutions are the bireal numbers $(x_1,0)$ and $(x_2,0)$. If the inequality $\geq$, in the previous condition, replaced by the inequality $<$, then the solutions are the bireal numbers (x_0,y_1) and (x_0,y_2). Let $a_1c_2 = a_2c_1$ and $a_2b_1 = a_1b_2$. If the condition $b_2^2 \geq 4a_2c_2$ ($b_1^2 \geq 4a_1c_1$), to which the second condition in (24) is reduced, is satisfied, then the solutions of *Hamilton's* general quadratic equation are the bireal numbers $(x_1,0)$ and $(x_2,0)$. If $a_1c_2 = a_2c_1$ and $a_1b_1 = -a_2b_2$, and $b_1^2 \geq -4a_2c_2$ ($b_2^2 \geq -4a_1c_1$) the solutions are bireal numbers $(0,y_1)$ and $(0,y_2)$.

As emphasized above, real solutions of the equation $ax^2 + bx + c = 0$ exist if the discriminant $\Delta = b^2 - 4ac$ is nonnegative. In addition, the solutions of the quadratic equation (12), over the field of bireal numbers $\mathbb{R}^2$, which also includes *Hamilton's* couple equation (2), are in the form of bireal numbers. Accordingly, in addition to the fact that there is a complete analogy between the solution of the quadratic equation (12) and the quadratic equation with complex coefficients, the question arises: Whether it was necessary to introduce the imaginary unit i at all? *Hamilton* himself emphasized the same thing in (Hamilton, 1853), but he did not insist on it, so it was not sufficiently noticed by the rest of scientific public. More precisely, an ordinary quadratic equation $ax^2 + bx + c = 0$, defined over the field of real numbers $\mathbb{R}$, either has real solutions (if Δ is nonnegative) or has no solutions. On the other hand, *Hamilton's* couple equation, being defined over a field of bireal numbers, decomposes into a system of two nonlinear equations with two unknowns x and y. The ordered pairs of solutions of the system are bireal numbers. Consequently, the equation $x^2 = -1$ has no solution in the field of real numbers, but the solutions of *Hamilton's* couple equation $w^2 = (x,y)^2 = -1$, which decomposes into a system of two nonlinear equations: $x^2 - y^2 = -1$ and $2xy = 0$, has two real solutions $(0,1)$ and $(0,-1)$, which belong to the field of bireal numbers.

GAUSSIAN ELIMINATION METHOD FOR SNAE

By decomposing *Hamilton's* quadratic equation (11), the following SNAE

$$\begin{aligned} a_{11}x^2 + a_{12}y^2 + 2b_1xy + c_{11}x + c_{12}y + d_1 = 0\\ a_{21}x^2 + a_{22}y^2 + 2b_2xy + c_{21}x + c_{22}y + d_2 = 0.\end{aligned} \tag{25}$$

is obtained. However, *Hamilton's* quadratic equation (11) can also be written in matrix form

$$\begin{bmatrix}\mathbf{1} & \mathbf{1}\end{bmatrix}\left(A\begin{bmatrix}x^2\mathbf{1}\\ y^2\hat{\mathbf{1}}\end{bmatrix}+B\begin{bmatrix}xy\mathbf{1}\\ xy\hat{\mathbf{1}}\end{bmatrix}+C\begin{bmatrix}x\mathbf{1}\\ y\hat{\mathbf{1}}\end{bmatrix}+\begin{bmatrix}d_1\mathbf{1}\\ d_2\hat{\mathbf{1}}\end{bmatrix}\right)=(0,0), \tag{26}$$

where

$$A=\begin{bmatrix} a_{11}\mathbf{1} & -a_{12}\hat{\mathbf{1}} \\ a_{21}\hat{\mathbf{1}} & a_{22}\mathbf{1}\end{bmatrix}=\begin{bmatrix}(a_{11},0) & (0,-a_{12}) \\ (0,a_{21}) & (a_{22},0)\end{bmatrix},\ C=\begin{bmatrix} c_{11}\mathbf{1} & -c_{12}\hat{\mathbf{1}} \\ c_{21}\hat{\mathbf{1}} & c_{22}\mathbf{1}\end{bmatrix}=\begin{bmatrix}(c_{11},0) & (0,-c_{12}) \\ (0,c_{21}) & (c_{22},0)\end{bmatrix} \text{ and}$$

$$B=\begin{bmatrix} b_1\mathbf{1} & -b_1\hat{\mathbf{1}} \\ b_2\hat{\mathbf{1}} & b_2\mathbf{1}\end{bmatrix}=\begin{bmatrix}(b_1,0) & (0,-b_1) \\ (0,b_2) & (b_2,0)\end{bmatrix}.$$

Since $\begin{bmatrix} x^2\mathbf{1} \\ y^2\hat{\mathbf{1}}\end{bmatrix}=\begin{bmatrix} x\mathbf{1} & 0\hat{\mathbf{1}} \\ 0\hat{\mathbf{1}} & y\mathbf{1}\end{bmatrix}\begin{bmatrix} x\mathbf{1} \\ y\hat{\mathbf{1}}\end{bmatrix}$ and $\begin{bmatrix} xy\mathbf{1} \\ yx\hat{\mathbf{1}}\end{bmatrix}=\begin{bmatrix} y\mathbf{1} & 0\hat{\mathbf{1}} \\ 0\hat{\mathbf{1}} & x\mathbf{1}\end{bmatrix}\begin{bmatrix} x\mathbf{1} \\ y\hat{\mathbf{1}}\end{bmatrix}$

it follows that

$$\begin{aligned}&[\mathbf{1}\quad \mathbf{1}](A\begin{bmatrix} x^2\mathbf{1} \\ y^2\hat{\mathbf{1}}\end{bmatrix}+B\begin{bmatrix} xy\mathbf{1} \\ xy\hat{\mathbf{1}}\end{bmatrix}+C\begin{bmatrix} x\mathbf{1} \\ y\hat{\mathbf{1}}\end{bmatrix}+\begin{bmatrix} d_1\mathbf{1} \\ d_2\hat{\mathbf{1}}\end{bmatrix})= \\ &=[\mathbf{1}\quad \mathbf{1}]((A\begin{bmatrix} x\mathbf{1} & 0\hat{\mathbf{1}} \\ 0\hat{\mathbf{1}} & y\mathbf{1}\end{bmatrix}+B\begin{bmatrix} y\mathbf{1} & 0\hat{\mathbf{1}} \\ 0\hat{\mathbf{1}} & x\mathbf{1}\end{bmatrix}+C)\begin{bmatrix} x\mathbf{1} \\ y\hat{\mathbf{1}}\end{bmatrix}+\begin{bmatrix} d_1\mathbf{1} \\ d_2\hat{\mathbf{1}}\end{bmatrix})=(0,0).\end{aligned} \tag{27}$$

Accordingly, $D\begin{bmatrix} x\mathbf{1} \\ y\hat{\mathbf{1}}\end{bmatrix}+\begin{bmatrix} d_1\mathbf{1} \\ d_2\hat{\mathbf{1}}\end{bmatrix}=\begin{bmatrix}(0,0) \\ (0,0)\end{bmatrix}$, where

$$D=\begin{bmatrix}(a_{11}x+b_1y+c_{11},0) & -(0,a_{12}y+b_1x+c_{12}) \\ (0,a_{21}x+b_2y+c_{21}) & (a_{22}y+b_2x+c_{22},0)\end{bmatrix}.$$

On the assumption that the square matrix $\underline{D}=\begin{bmatrix} a_{11}x+b_1y+c_{11} & a_{12}y+b_1x+c_{12} \\ a_{21}x+b_2y+c_{21} & a_{22}y+b_2x+c_{22}\end{bmatrix}$, such that $|D|=|\underline{D}|\mathbf{1}$, is a regular matrix, one obtains that

$$\begin{aligned}\begin{bmatrix} x\mathbf{1} \\ y\hat{\mathbf{1}}\end{bmatrix}&=-D^{-1}\begin{bmatrix} d_1\mathbf{1} \\ d_2\hat{\mathbf{1}}\end{bmatrix}=-\frac{adjD}{|D|}\begin{bmatrix} d_1\mathbf{1} \\ d_2\hat{\mathbf{1}}\end{bmatrix}=-\frac{\begin{bmatrix}(a_{22}y+b_2x+c_{22},0) & (0,a_{12}y+b_1x+c_{12}) \\ -(0,a_{21}x+b_2y+c_{21}) & (a_{11}x+b_1y+c_{11},0)\end{bmatrix}}{|\underline{D}|}\begin{bmatrix} d_1\mathbf{1} \\ d_2\hat{\mathbf{1}}\end{bmatrix}= \\ &=\frac{\begin{bmatrix}[(b_1d_2-b_2d_1)x+(a_{12}d_2-a_{22}d_1)y+(c_{12}d_2-c_{22}d_1)]\mathbf{1} \\ [(a_{21}d_1-a_{11}d_2)x-(b_1d_2-b_2d_1)y+(c_{21}d_1-c_{11}d_2)]\hat{\mathbf{1}}\end{bmatrix}}{(a_{11}x+b_1y+c_{11})(a_{22}y+b_2x+c_{22})-(a_{21}x+b_2y+c_{21})(a_{12}y+b_1x+c_{12})}.\end{aligned} \tag{28}$$

On the basis of the previous equation, it follows that

$$\begin{aligned}[|\underline{D}|-(b_1d_2-b_2d_1)]xy&=(a_{12}d_2-a_{22}d_1)y^2+(c_{12}d_2-c_{22}d_1)y= \\ &=(a_{21}d_1-a_{11}d_2)x^2-2(b_2d_1-b_1d_2)xy+(c_{21}d_1-c_{11}d_2)x,\end{aligned} \tag{29}$$

where $|\underline{D}| = (a_{11}b_2 - a_{21}b_1)x^2 + (a_{22}b_1 - a_{12}b_2)y^2 + |A|xy + (a_{11}c_{22} - a_{21}c_{12})x + (a_{22}c_{11} - a_{12}c_{21})y + |C|$, $|A| = a_{11}a_{22} - a_{12}a_{21}$ and $|C| = c_{11}c_{22} - c_{12}c_{21}$. The solutions x and y of SNAE (25) belong to the set $\{x,y,c\}$ of solutions of the following SNAE

$$\begin{gathered}[|\underline{D}| - (b_1d_2 - b_2d_1)]xy = c\\ (a_{12}d_2 - a_{22}d_1)y^2 + (c_{12}d_2 - c_{22}d_1)y - c = 0\\ (a_{21}d_1 - a_{11}d_2)x^2 - [2(b_1d_2 - b_2d_1)y - (c_{21}d_1 - c_{11}d_2)]x - c = 0,\end{gathered} \tag{30}$$

where c is an arbitrary real constant, which must satisfy two conditions. The first condition is that the discriminant Δ_y of the second equation of the system (30) is nonnegative. In that case, solving the second equation leads to the functional solution $y = f_y(c)$. The second condition is that the discriminant Δ_x of the third equation of the system (30), which includes the previous functional solution, must also be nonnegative. Under that condition and solving that equation, leads to another functional solution $x = f_x(c)$. By this elimination method of solving SNAE (25), and which is obviously similar to solving a system of linear algebraic equations by the *Gaussian* elimination method, in special cases, as in the example that follows, the solution for the unknown c, of the first equation of system (30), which includes functional solutions $f_y(c)$ and $f_x(c)$, can be obtained in a relatively simple way. However, in the general case, due to the complexity of the algebraic equation itself, in order to obtain a solution for the unknown c, computational mathematics methods would have to be used to numerically solve algebraic equations.

Example: SNAE (21): $x^2 - y^2 = d$ and $2xy = d/b$, is obtained from SNAE (25) for $a_{21} = a_{22} = b_1 = c_{11} = c_{12} = c_{21} = c_{22} = 0$, $b_2 = b \neq 0$, $a_{11} = -a_{12} = 1$ and $d_1 = d_2 = -d \neq 0$. In that case, according to (30), solutions x and y of this SNAE are solutions of SNAE: $(|\underline{D}| - bd)xy - c = 0$, $x^2 - 2byx - c/d = 0$ and $y^2 - c/d = 0$. From the last equation of this system it follows that $y = \pm\sqrt{c/d}$. If this value for y is included in the second equation of the system, the value for x is obtain, so that $x = (b \pm \sqrt{1+b^2})y$. On the other hand,

$$|\underline{D}| = b(x^2 + y^2) = \frac{bc}{d}[1 + (b \pm \sqrt{1+b^2})^2] = \frac{bc}{d}(1 + \frac{b \pm \sqrt{1+b^2}}{-b \pm \sqrt{1+b^2}}) = \frac{2bc}{d}\frac{\pm\sqrt{1+b^2}}{-b \pm \sqrt{1+b^2}}.$$

According to the first equation of the sistem,

$$\frac{|\underline{D}|}{d} = \frac{1}{b \pm \sqrt{1+b^2}} + b = \pm\sqrt{1+b^2},$$

since $xy = c(b \pm \sqrt{1+b^2})/d$. Therefore, $2bc = d^2(-b \pm \sqrt{1+b^2}) \geq 0$, so that

$$y = \pm\sqrt{\frac{d}{2b}(-b \pm \sqrt{1+b^2})} \text{ and } x = (b \pm \sqrt{1+b^2})y = \pm\sqrt{\frac{d}{2b}(b \pm \sqrt{1+b^2})}.$$

CONCLUSIONS

The main purpose of this manuscript is to reaffirm *Hamilton's* idea to make the solutions of polynomial equations completely real and autonomous from complex numbers (Hamilton, 1853). Namely, by defining couple (bireal) numbers, *Hamilton* reduces the system of two algebraic equations with two unknowns to one couple equation, the solutions of which are ordered pairs of real numbers. In addition, the so called matrix method for solving *Hamilton's* quadratic equations is presented, which in the general case is somewhat analogous to the *Gaussian* elimination method. Finally, it was presented that *Hamilton's* linear differential equation, with a system of two characteristic algebraic equations, also has solutions in the form of an ordered pair of two real numbers. Accordingly, we can draw the conclusion that bireal numbers, as solutions of polynomial equations, are more suitable and mathematically correct than complex numbers.

DECLARATIONS OF INTEREST STATEMENT

The authors affirm that there are no conflicts of interest to declare in relation to the research presented in this paper.

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